Valuation Bounds of Tranche Options

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Abstract

We performed a comprehensive analysis on the price bounds of CDO tranche options, and illustrated that the CDO tranche option prices can be effectively bounded by the joint distribution of default time ($JDDT$) from a default time copula. Systemic and idiosyncratic factors beyond the $JDDT$ only contribute a limited amount of pricing uncertainty. The price bounds of tranche option derived from a default time copula are often very narrow, especially for the senior part of the capital structure where there is the most market interests for tranche options. The tranche option bounds from a default time copula can often be computed semi-analytically without Monte Carlo simulation, therefore it is feasible and practical to price and risk manage senior CDO tranche options using the price bounds from a default time copula only.

CDO tranche option pricing is important in a number of practical situations such as counterparty, gap or liquidation risk; the methodology described in this paper can be very useful in the above described situations.

1 Introduction

The credit derivative market has experienced tremendous volatility since the beginning of the sub-prime and credit crisis. The standard credit index swaps and index tranches have become very important instruments for market participants to hedge or take positions on the overall credit quality and credit correlation. The index swaption market has become more active recently because of the increasing need to manage the volatility of market-wide credit movements. On the other hand, the index tranche option market never gained any traction despite the large realized volatility in the index tranches. The reasons are three fold: first, the index tranches are less liquid than the index swaps, secondly the standard index tranches can be viewed as an option on the index portfolio loss and it already provides leverages, therefore there is no need for investors to trade index tranche options in order to get leveraged exposure; thirdly, there is no standard model that can price and hedge index tranche options. It remains a very challenging modelling problem to properly price CDO tranche options and the market participants generally lack the confidence in pricing and hedging the index tranche options. Despite the lack of interest to trade tranche options directly, it is very important to study the valuation of tranche options since they naturally arise from a number of common practical situations, for example in conterparty risk, gap risk or liquidation risk.

Under current market conditions, it is almost impossible to price tranche options precisely because of the lack of relevant market observables. (Mashal & Naldi 2005) suggested a method to compute the range

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bounds of tranche options from a default time copula. The key contribution of that paper is a scheme to compute the range bounds without the nested Monte Carlo simulation, which leads to easy calculations of the price bounds implied by a default time copula via a regular Monte Carlo simulation.

A default time copula, by definition, only models the joint distribution of default time ($JDDT$), and it does not model any other factors. On the other hand, a bottom-up dynamic spread model attempts to model the joint distribution of default time and all the systemic and idiosyncratic factors that affect the spread dynamics. Dynamic factor model is the most common approach to build a bottom-up dynamic spread model, where the spread dynamics are driven by a few systemic factors that affects all the names and an idiosyncratic factor for each individual name. The idiosyncratic factors are easy to model because they are independent from other factors by definition, therefore the main task of building a dynamic factor model is to construct the joint distribution of default time and systematic factors ($JDDTSF$) which fully specifies the systemic dynamics.

The dynamic factor model is more difficult to build and calibrate than a default time copula since it needs to model more factors beyond the default time. The methodology prescribed by (Mashal & Naldi 2005) can be very useful in practice if the resulting option price bounds from the default time copula are narrow; as it allow us to price and risk manage tranche options without implementing a full dynamic factor model. (Mashal & Naldi 2005) have shown that the price bounds of tranche options from a standard Gaussian Copula model are very narrow; however it is unclear if the price bounds would remain narrow in a more realistic situation where the default time copula has to be calibrated to the index tranche market across multiple maturities.

Recently, we suggested a very flexible dynamic correlation modelling framework in (Li 2009). A key finding of that paper is that the portfolio loss distribution and CDO tranche prices only depends on the joint distribution of default indicators ($JDDI$); the modelling framework is more flexible than previous bottom-up models in the literature as it allows the $JDDT$ and $JDDTSF$ to change independently from the $JDDI$. With (Li 2009), once we calibrated the $JDDI$ to index tranche prices, we can easily construct different default time copula or dynamic factor models without changing the calibrated index tranche prices. In this study, we used the (Li 2009) model to construct different default time copulas from the same index tranche calibration, and we systematically study the price bounds of tranche options under these default time copulas.

The paper is organized as follows: In section 2 we first review some practical situations that involve the pricing of tranche options; then we review the general derivation of the price bounds for tranche options in section 3; then we study the pricing bounds for European style tranche options in section 4; then we discuss tranche options with random triggers in section 5.

Even though we focus on tranche options in this paper, the methodology and conclusions are generally applicable to other types of multi-name credit options, such as options on NtD basket, and options on multiple tranches or CDO^{2}s.

2 Practical Examples of Tranche Options

Tranche options naturally arise from a number of practical situations. We first define some terminology before reviewing these situations. Suppose there exists a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ equipped with a risk-neutral probability measure $\mathbb{P}$. Consider a CDO tranche with a fixed set of payment date $\{t_i\}$ and a stream of cashflow $\{c_i\}$ on the payment grid. The MTM of the tranche at time t is $V_t = B_t \mathbb{E}[\sum_{t_i > t} \frac{c_i}{B_i} | \mathcal{F}_t]$, where B_i is value of a money market account at t_i that started with amount 1 at time 0. We further assume that B_i and

c_i are uncorrelated, therefore:

$$V_t = \sum_{t_i > t} d(t,t_i)\mathbb{E}[c_i|\mathcal{F}_t]$$

where $d(t,t_i) = \frac{B_t}{\mathbb{E}[B_{t_i}|\mathcal{F}_t]}$ is the risk free discount factor between t and t_i.

2.1 Counterparty Risk

This is the classic case considered in (Mashal & Naldi 2005). Suppose a bank traded a tranche with a risky counterparty, if the counterparty default at time t, then the bank usually need to pay the full MTM (V_t) to the bankruptcy pool if the trade is to the counterparty's favor ($V_t < 0$), and the bank only recover a portion of the MTM if the trade's MTM is to the bank's favor ($V_t > 0$). Define R as the recovery rate of the counterparty, then the bank could suffer a loss if the counterparty default, and the amount of the lose is $(1-R)\max[V_t, 0]$, which is a typical call option payoff. The counterparty effectively holds a call option to default and walk away from the remaining trade. The price of this call option to the counterparty is therefore:

$$\mathrm{CP} = \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max[(1-R)V_t, 0]]$$

The fair MTM price of the trade to the bank therefore has to be adjusted down to $V_0 - \mathrm{CP}$ if we price in the counterparty default risk. In reality, the bank also holds an option to default, which can be priced similarly.

2.2 Gap Risk

Suppose a bank entered a tranche trade with a client, and the client posted collateral in the amount of C_0 according to certain margin policies. Normally the collateral agreement allows the bank to make a margin call for additional collateral if the market moves against the client and the initial margin is inadequate to cover the potential loss of the trade. Suppose V_t is the MTM to the bank, and a margin call is made at time t, if the client does not post additional collateral within a certain time period δ, which is typically a few days to two weeks after the margin call, the bank can seize the collateral and unwind the trade. A rationale client would choose not to post any additional collateral in the event of $C_0 < V_{t+\delta}$, therefore the client effectively holds a call option whose payoff is $\max(V_{t+\delta} - C_0, 0)$. Denote τ as the stopping time of the margin call, then the price of the call option to the client is:

$$\mathrm{GAP} = \mathbb{E}[d(0,t+\delta)\mathbf{1}_{\tau=t}\max(V_{t+\delta} - C_0, 0)]$$

The fair price of the instrument to the bank with this gap risk is therefore $V_0 - \mathrm{GAP}$.

2.3 Levered Super Senior Tranche

Levered super senior (LSS) trade is a very popular trade for a client to take on leveraged risk on the senior part of the capital structure. In a typical LSS trade, the client sell protection on a super senior tranche V_t to a bank and the client only post collateral in the amount of C_0. The ratio between the notional amount of the V_t and C_0 is the leverage factor. The LSS trade is different from the situation in the gap risk in that the bank can only call for additional collateral if a pre-defined trigger event occurs. The trigger event can be portfolio loss reaching certain level, or tranche spreads reaching certain level. Before the trigger event, the bank can't call additional collateral beyond C_0 even if the market moves against the client and the MTM of V_t to the bank becomes greater than the collateral value C_0. After the trigger event, the bank usually is free to call additional collateral based on the MTM of the super senior tranche V_t.

Since a rational client would not post any additional collateral if $C_0 < V_t$, the bank only gets the smaller of C_0 and V_t when trigger event occurs, therefore the value of the LSS trade to the bank is:

$$\begin{aligned}
\text{LSS} &= \mathbb{E}[\sum_i d(0,t_i)\mathbf{1}_{t_i<\tau} c_i + d(0,t)\mathbf{1}_{\tau=t} \min(V_t, C_0)] \\
&= \mathbb{E}[\sum_i d(0,t_i)\mathbf{1}_{t_i<\tau} c_i + d(0,t)\mathbf{1}_{\tau=t} (V_t - \max(V_t - C_0, 0)] \\
&= \mathbb{E}[\sum_i d(0,t_i)\mathbf{1}_{t_i<\tau} c_i + d(0,t)\mathbf{1}_{\tau=t} V_t] - \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t} \max(V_t - C_0, 0)]
\end{aligned}$$

where c_i is the coupon payment for the super senior tranche, here we made the assumption that the trigger event always occurs before the super senior tranche suffers any real losses, which is almost always the case in practice since the trigger is put in to protect the bank thus it is designed to trigger far before the realized loss hits the tranche attachment. If the trigger is based on the portfolio loss, the first term can be computed from a default time copula. The second term is the value of a call option for the client to walk away from the trade when the trigger event occurs.

The LSS trade is often mistakenly modeled as a gap risk trade. Comparing the LSS trade with the gap risk, it is obvious that the LSS protection worths much less to the bank than in the case of the gap risk. Readers are referred to (Gregory 2008) for a very detailed discussion of the LSS.

2.4 Liquidation Risk

Suppose a client entered a funded credit-linked-note (CLN) trade referencing a tranche with a bank. At trade inception, the client deposits the face amount of the CLN to a SPV, this principal may be invested in a risky asset A_t for additional yield. The SPV then enters a unfunded swap contract with the bank to get exposure to the underlying tranche. We denote the MTM of the unfunded swap to the bank as V_t. The coupon payments from both the A_t and V_t, netting of any fees to the bank, will be paid to the client. If either the underlying tranche gets impaired, or if the risky collateral A_t defaults, both the V_t and A_t are liquidated; the client then receives the liquidation value of $(A_t - V_t)$ if it is positive. Since the client never put additional money into the SPV besides the initial principal, the bank will suffer a loss in the event of the net liquidation value $A_t - V_t$ is negative. Define τ to be the stopping time of the liquidation event, then the client effectively holds a liquidation option whose payoff is $\max(V_t - A_t, 0)$ whose value to the client is:

$$\text{LIQ} = \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t} \max(V_t - A_t, 0)]$$

The fair value of the swap to the bank is therefore $V_0 - \text{LIQ}$. The liquidation risk is similar to an exchange option between two assets V_t and A_t.

There are several variations of the liquidation risk: in the famous (or infamous) mini-bond structure, V_t is a first-to-default basket and A_t is a synthetic CDO tranche. In a typical credit-linked-note, the V_t can be a single name CDS or synthetic CDO tranche, and A_t is very safe money market instruments. In a funding trade, the V_t can be a CDO tranche or a single name CDS, and the A_t is the term bond or funding of the bank.

2.5 Callable Tranche

Suppose a bank bought tranche protection from a client, and the trade's MTM to the bank is V_t. If the client is given the option to buy back the tranche protection at price K under certain trigger event, then the client hold an option of $\max(V_t - K, 0)$ when the trigger event occurs. Denote τ as the stopping time of the trigger event, the client's option to call the tranche can be valued as:

$$\text{CAL} = \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t - K, 0)]$$

The fair value of the swap to the bank is therefore $V_0 - \text{CAL}$.

3 Derivation of Price Bounds

All of examples in section 2 reduces to the same problem of valuing the following call option where the exercise time is random:

$$C = \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t - K, 0)] \tag{1}$$

with K as the strike price of the call. V_t may involve multiple tranches or assets as in the case of liquidation risk. This call option is very difficult to price because it depends on the MTM V_t at a future time. Normally this types of problem requires nested Monte Carlo simulation because V_t itself is an expectation of all future cashflows. It also requires a full dynamic model capable of generating spread levels at a future time on a simulated path, such a model is nearly impossible to calibrate given the lack of liquidity in the tranche option market. (Mashal & Naldi 2005) offered an elegant solution to compute the range bounds of the option value C just from the filtration generated by default events and recovery rates only (denoted as $\mathcal{D}_t$). Note that the full market filtration $\mathcal{F}_t$ also include other systemic and idiosyncratic factors beyond $\mathcal{D}_t$, therefore $\mathcal{D}_t \subset \mathcal{F}_t$. We also have to assume that the trigger event τ is adapted to $\mathcal{D}_t$, which excludes the spread triggers.

In this section, we review the derivation of the range bounds of (1). The upperbound of (1) is the same method as described in (Mashal & Naldi 2005), the lowerbound of (1) given here is an improvement over the method in (Mashal & Naldi 2005), which first appeared in (Ruan 2006). The key in the derivation is the Jensen's inequality which states: $g(\mathbb{E}[x]) \leq \mathbb{E}[g(x)]$ if $g(x)$ is a convex function. In particular, since $\max(x-K,0)$ is a convex function of x: $\max(\mathbb{E}[x]-K,0) \leq \mathbb{E}[\max(x-K,0)]$.

Recall that $V_t = \mathbb{E}[\sum_{t_i>t} d(t,t_i)c_i|\mathcal{F}_t]$ where c_i are the cashflows of the trade. c_i is assumed to be adapted to $\mathcal{D}_{t_i}$, which is usually the case in practice, i.e., the cashflows of multi-name credit derivatives normally are only functions of realized default and recovery scenarios. The upper bound of C can be derived as:

$$\begin{aligned}
C &= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t - K, 0)] && \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\mathbb{E}[\sum_{t_i>t} d(t,t_i)c_i|\mathcal{F}_t] - K, 0)] && : \text{expand } V_t \\
&\leq \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\mathbb{E}[\max(\sum_{t_i>t} d(t,t_i)c_i - K, 0)|\mathcal{F}_t]] && : \text{Jensen's inequality} \\
&= \mathbb{E}[\mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\sum_{t_i>t} d(t,t_i)c_i - K, 0)|\mathcal{F}_t]] && : \mathbf{1}_{\tau=t} \text{ is adapted to } \mathcal{F}_t \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\sum_{t_i>t} d(t,t_i)c_i - K, 0)] && : \text{iterative expectation}
\end{aligned} \tag{2}$$

It is very straight-forward to compute the upper bound from a Monte Carlo simulation of default times and recovery rates since there is no nested simulations. Suppose $\mathcal{Y}_t$ is a sub filtration of $\mathcal{F}_t$ that includes the trigger event. The lower bound of C can be derived as:

$$
\begin{aligned}
C &= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t-K,0)] \\
&= \mathbb{E}[\mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t-K,0)|\mathcal{Y}_t]] && \text{: iterative expectation} \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\mathbb{E}[\max(V_t-K,0)|\mathcal{Y}_t]] && \text{: } \mathbf{1}_{\tau=t} \text{ is adapted to } \mathcal{Y}_t \\
&\geq \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\mathbb{E}[V_t-K|\mathcal{Y}_t],0)] && \text{: Jensen's inequality} \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\mathbb{E}[V_t|\mathcal{Y}_t]-K,0)] && \text{: K is constant} \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\mathbb{E}[\mathbb{E}[\sum_{t_i>t} d(t,t_i)c_i|\mathcal{F}_t]|\mathcal{Y}_t]-K,0)] && \text{: expand } V_t \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\mathbb{E}[\sum_{t_i>t} d(t,t_i)c_i|\mathcal{Y}_t]-K,0)] && \text{: iterative expectation}
\end{aligned} \tag{3}
$$

The term $\mathbb{E}[\sum_{t_i>t} d(0,t_i)c_i|\mathcal{Y}_t]$ is the expected total value of future cashflow conditioned on the information in $\mathcal{Y}_t$. The choice of $\mathcal{Y}_t$ determines the quality of the lower bound, the more information in $\mathcal{Y}_t$ the higher the lower bound is. In the limiting case of $\mathcal{Y}_t = \mathcal{F}_t$, the lower bound converges to the true value of the option.

There is a very intuitive explanation of the upper and lower bounds of the option values. The MTM of the underlying tranche (V_t) is based on the information in the market filtration $\mathcal{F}_t$. In (2), the upper bound of option payoff $\mathbf{1}_{\tau=t}\max(\sum_{t_i>t} d(0,t_i)c_i,0)]$ corresponds to the option's value to an all-powerful deity who can perfectly foresee the future default events and recovery rates. Therefore, at the time of the trigger event $\tau = t$, the deity will exercise the option based on the foreseeable future cashflow $\sum_{t_i>t} d(0,t_i)c_i$ instead of V_t. For example, the deity may exercise the option and buy protection on a CDO tranche even if the MTM of the tranche is less than the strike price because he foresees that the future loss of the tranche will eventually exceed the future value of the strike price. Therefore, the deity can extract more value from the option than its fair market value by exercising the option based on future information that is not part of $\mathcal{F}_t$. Therefore, the upper bound corresponds to the option value with the divine power of perfect foresight.

The lower bound of the option payoff $\mathbf{1}_{\tau=t}\max(\mathbb{E}[\sum_{t_i>t} d(0,t_i)c_i|\mathcal{Y}_t],0)$ corresponds to an imprisoned investor who were only given the information in the sub-filtration $\mathcal{Y}_t \subset \mathcal{F}_t$. Therefore, he does not observe the fair MTM value V_t, and he can only exercise the option based on $\mathbb{E}[V_t|\mathcal{Y}_t]$, which is an estimation (or best guess) of the MTM based on the available information $\mathcal{Y}_t$. This clearly results in suboptimal exercise of the option. Therefore, the lower bound corresponds to the option value with incomplete information.

Even with a full dynamic model, one could obtain the lower bound of the option instead of the true value if the numerical methods of the option pricing is built on a reduced filtration, which is often the case with the lattice methods. For example, in the pioneering work by (Chapovsky, Rennie & Tavares 2006), the tranche options are priced by building a lattice on a reduced filtration with low dimensionality, which results in a lower bound of the option in the strict sense even though the authors were trying to obtain the true value of the option.

In the most generic form, the upper bound and lower bound can be computed from the Monte Carlo simulation of a default time copula. We don't need to model or simulate any future spreads in order to compute the price bounds if we choose $\mathcal{Y}_t \subset \mathcal{D}_t$. The upper bound can be computed directly from the simulated default time and recoveries of all the underlying credits, and the lower bound requires a least square Monte Carlo simulation as in (Longstaff & Schwartz 2001) that regresses the value of the future cash flows to the state variables in $\mathcal{Y}_t$. Semi-analytical solution to the option bounds can be obtained if the trigger event is deterministic in time (ie, vanilla European option whose holder can exercise at a deterministic future time), or if the trigger event is the default event of a single credit, such as the case in the counterparty risk. We'll analyze these special cases in the following sections.

The bounds derived from the default time copula offers great insights on the tranche option pricing. If the bounds are very narrow, the option values are mainly determined by the $JDDT$; if the bounds are wide, then the option values are primarily determined by other systemic or idiosyncratic factors beyond the default time. In this paper, we'll try to understand what is the main driver of the tranche option prices.

The price bounds for put option can be obtained via the put-call parity. The methodology to obtain the lower and upper bound of an option payoff is very general, it applies to any multi-name credit options, such as CDO tranche option, NTD basket option or the multiple asset options as discussed in liquidation risk.

4 European Tranche Options

In this section, we consider the vanilla European tranche options which can be exercised at a pre-determined time. The tranche options with random trigger event will be discussed in the next section.

For simplicity, we consider the tranche loss option instead of the more general case of tranche option. An European tranche loss option is a hypothetical instrument that gives the buyer the right (not an obligation) to pay a fixed amount K at the exercise time t in order to receive a payoff equal to the total realized loss of a tranche at time T. Regular tranche option reduces to the tranche loss option if the tranche has no running coupon and if we assume the protection payments are all made at maturity instead of the time of default. The price bounds of tranche loss option with deterministic time trigger can be computed without Monte Carlo simulation. Since the main driver of a tranche's value is its expected tranche loss at maturity, the conclusions drew from the analysis on the tranche loss option applies to the more general cases of tranche option with running coupons and immediate protection settlement. Furthermore, the regular tranche option can be approximated using tranche loss option, please see Appendix A for a more detailed discussion of the approximation.

Then the PV of the tranche loss option can be written as:

$$\begin{aligned} C &= d(0,t)\mathbb{E}[\max(d(t,T)\mathbb{E}[L_T(A,D)|\mathcal{F}_t] - K, 0)] \\ &= d(0,T)\mathbb{E}[\max(\mathbb{E}[L_T(A,D)|\mathcal{F}_t] - \frac{K}{d(t,T)}, 0)] \end{aligned} \tag{4}$$

We use A and D to denote the tranche's attachment and detachment levels. In the subsequent analysis, we drop all the deterministic discount factors to simplify the exposition, with the understanding that the valuation bounds and strikes need to be adjusted with those deterministic discount factors in (4). The $L_T(A,D)$ in (4) is the expected tranche loss (ETL) and L_T is the portfolio loss at tranche maturity T:

$$L_T(A,D) = \min(\max(L_T - A, 0), D - A)$$

A simple expression for the upperbound can be derived from (2):

$$\begin{aligned} C &\leq \mathbb{E}[\max(L_T(A,D) - K, 0)] \\ &= \mathbb{E}[\max(\min(\max(L_T - A, 0), D - A) - K, 0)))] \\ &= \mathbb{E}[\min(\max(L_T - (A+K), 0), D - (A+K))] \\ &= \mathbb{E}[L_T(A+K, D)] \end{aligned} \tag{5}$$

Therefore, the upper bound of the tranche loss option is just the ETL of an $A+K$ to D tranche. This relationship does not hold if the tranche has a non-zero running coupon or if the protection payment is not made at the end of tranche maturity. However, since the impact of running coupon and the discounting of

Table 1: CDX-IG9 Expected Tranche Loss

Tranches	3Y	5Y	7Y	10Y
0-3%	54.12%	80.19%	86.76%	91.12%
3-7%	17.03%	42.64%	55.16%	66.18%
7-10%	5.36%	20.09%	33.98%	48.18%
10-15%	1.35%	8.17%	15.82%	23.34%
15-30%	0.76%	2.29%	4.81%	7.95%
30-60%	0.49%	1.62%	3.40%	5.31%
60-100%	0.02%	0.42%	0.95%	1.54%

protection payments is limited in the tranche pricing, we can still use the PV of a $A+K$ to D tranche as an approximation to the upper bound of a regular tranche option. This is a very handy relationship in practice. A more accurate upper bound for the regular tranche option can be obtained using the approximation in Appendix A.

The upper bound of the tranche loss option only depends on the terminal loss distribution, therefore, it is not model dependent as long as all the models are calibrated to the same loss distribution. For example, we can compute the upper bound of a tranche option even from a base correlation model.

The lower bound of the tranche loss option can be written as:

$$C \geq \mathbb{E}[\max(\mathbb{E}[L_T(A,D)|\mathcal{Y}_t] - K, 0)] \tag{6}$$

where $\mathcal{Y}_t$ is a sub-filtration of the market filtration $\mathcal{F}_t$. The lower bound is generally model dependent through the conditional expectation $\mathbb{E}[L_T(A,D)|\mathcal{Y}_t]$ but we can derive a naive model-independent lower bound by Jensen's inequality:

$$\begin{aligned} C &= \mathbb{E}[\max(\mathbb{E}[L_T(A,D)|\mathcal{F}_t] - K, 0)] \\ &\geq \max(\mathbb{E}[\mathbb{E}[L_T(A,D)|\mathcal{F}_t] - K], 0) \\ &= \max(\mathbb{E}[L_T(A,D)] - K, 0) \end{aligned} \tag{7}$$

The lower bound in (7) corresponds to an exercising strategy that the option holder always exercises the option if the expected tranche loss based on information at $t=0$ is more than the strike price K, aka, if the option is "in-the-money" at $t=0$. (Gregory 2008) pointed out that a digital tranche is the upper bound ("super-hedge") of a loss-trigger LSS trade. The digital tranche is equivalent to the naive lower bound (7) in the context of the LSS.

To get more precise lower bound, we have to choose the sub-filtration $\mathcal{Y}_t$ with more information. We used the CDX-IG9 index tranches and market data on Jul 21st, 2009 for this study. We calibrated the model described in (Li 2009) to the market data, and Table 1 showed the expected tranche loss from the calibrated model. Note that all the ETLs are normalized to their tranche notionals, so are the option values in the rest of this document[1].

We first consider IG9 tranche loss options that expires at 3Y (maturity: Dec. 20, 2010) for the expected loss of a 5Y (maturity: Dec. 20, 2012) tranche, we define the at-the-money (ATM) strike to be the expected tranche loss $K^{ATM} = \mathbb{E}[L_T(A,D)]$. In this study, we also computed the price bounds for in-the-money(ITM) and out-of-the-money (OTM) tranche loss options. In the following examples, the ITM strike is half of ETL, and the OTM strike is twice of the ETL: $K^{ITM} = \frac{1}{2}\mathbb{E}[L_T(A,D)]$ and $K^{OTM} = 2\mathbb{E}[L_T(A,D)]$. Table 2 showed the upper bounds computed from the calibrated bottom-up model according to (5).

[1]We still refer the CDX-IG9 tranches using their original strikes even though the actual calculations were using the adjusted strikes which take into account the three defaulted names in the portfolio.

Table 2: Upper Bounds of 3Y-5Y Tranche Loss Option

CDX-IG9	Upper Bounds		
Tranches	ITM	ATM	OTM
0-3%	43.33%	12.73%	0.00%
3-7%	30.71%	20.57%	4.44%
7-10%	17.35%	14.79%	10.21%
10-15%	7.63%	7.11%	6.14%
15-30%	2.24%	2.19%	2.10%
30-60%	1.61%	1.59%	1.56%
60-100%	0.42%	0.41%	0.41%

We now focus on the lower bounds which depends on the choice of the sub-filtration $\mathcal{Y}_t$.

4.1 Lower Bounds from Top-down Models

The minimum sub-filtration that can price tranche loss option consistently is the filtration generated by the portfolio loss process, we denote it as $\mathcal{L}_t$. Note that $\mathcal{L}_t$ does not contain any single name information and typical top-down models are built on the $\mathcal{L}_t$ filtration.

The $\mathcal{L}_t$ does impose a more precise lower bounds of the tranche loss option than (7). To illustrate this, we take a discrete sample of the initial loss distribution, and built two different Markov chains on the loss distribution: co-monotonic Markov chain and maximum entropy Markov chain. The details of how to build these Markov chains can be found in (Epple, Morgan & Schloegl 2007). Once we have a Markov chain on the loss transition, we can then compute the lower bound in (6) by conditioning on the portfolio loss L_t. Since the conditioning is only on a scaler variable, the lower bound can be easily computed from the Markov Chain without using Monte Carlo simulation. The lower bounds from the two different Markov chains are shown in the table 3: the co-monotonic Markov chain implies a much higher lower bound than the maximum entropy Markov chain. The OTM option on the 0-3% equity tranche has a value of 0 since the OTM strike is more than the tranche notional. The 60-100% tranche's lower bounds with co-monotonic Markov chain are slightly higher than the upper bounds in Table 2, which is caused by the inaccuracies of the discrete sampling of the loss distribution.

An interesting question is: what is the lower bound if we only know the loss distributions but not the Markov chain of loss transition? This bound is of special interest because it is not model dependent, and it is the lowest lower bound among all admissible Markov chains by the loss distribution. We denote this lowest lower bound as LLB($\mathcal{L}_t$). Finding the LLB($\mathcal{L}_t$) among all possible Markov chains can be formulated as a nonlinear optimization problem (see the Appendix B), which can be solved using a standard non-linear optimizer. The column "LLB" in table 3 is the lowest lower bound obtained from the nonlinear optimization. Note that the optimizations to find the LLBs for different tranches are run separately, therefore the tranche loss options from different tranches can't be at their LLB($\mathcal{L}_t$) simultaneously. For example, if the tranche loss option for the 0-3% tranche is priced at its LLB($\mathcal{L}_t$), then the 3-7% tranche loss option price has to be greater than its LLB($\mathcal{L}_t$) since the Markov chain that produces the LLB($\mathcal{L}_t$) for 0-3% tranche is generally not the same Markov chain that produces the LLB($\mathcal{L}_t$) of 3-7%. Though still crude, the LLB($\mathcal{L}_t$) from loss distribution is much more precise than the naive lower bound in (7), which are zeros for all the ATM or OTM options.

In table 3, the LLB($\mathcal{L}_t$) are greater than 0 for all tranches even in the case of OTM options. This is a unique feature of the CDO tranche option. In other asset classes, the OTM option values can be very close to

Table 3: Lower Bounds of 3Y-5Y Option from $\mathcal{L}_t$ (Top-down)

CDX-IG9	ITM Lower Bounds			ATM Lower Bounds			OTM Lower Bounds		
Tranches	Co-mo	Max-E	LLB	Co-mo	Max-E	LLB	Co-mo	Max-E	LLB
0-3%	43.09%	39.97%	39.97%	12.50%	7.91%	6.40%	0.00%	0.00%	0.00%
3-7%	30.32%	22.34%	21.40%	19.75%	11.74%	7.55%	4.18%	1.82%	1.54%
7-10%	17.01%	11.05%	9.82%	14.39%	7.53%	3.67%	9.35%	3.99%	2.28%
10-15%	7.63%	4.88%	4.20%	7.13%	3.64%	1.07%	6.28%	2.38%	0.94%
15-30%	2.19%	1.48%	1.12%	2.11%	1.24%	0.68%	2.02%	1.00%	0.66%
30-60%	1.60%	1.11%	0.81%	1.57%	0.97%	0.41%	1.52%	0.83%	0.40%
60-100%	0.48%	0.35%	0.28%	0.48%	0.30%	0.07%	0.47%	0.26%	0.03%

0 if the volatility of the underlying asset becomes very low. However, even the OTM tranche option always have certain minimum value regardless of the tranche spread volatility. The reason is that the dynamics of the portfolio loss process has to be consistent with the initial loss distribution at time $t = 0$, which imposes a minimum level of portfolio loss volatility. For example, the volatility of portfolio loss process cannot be 0 since a deterministic portfolio loss process clearly violates the initial loss distribution at $t = 0$.

The value of a tranche option depends on the full loss distribution hence it is important to model tranche options on the same underlying portfolio across capital structure as inter-dependent instruments. The simple approach of modeling tranche options as separate derivative instruments on individual tranches, as suggested by (Hull & White 2007), is not adequate. (Hull & White 2007) attempted to model tranche options using a similar approach to the Libor market model in the interest rates world, which could produce inconsistent prices with the underlying tranche prices. For example, there is no restriction on the volatility parameter of the forward tranche spread in the (Hull & White 2007) approach and we could produce arbitrage-able tranche option prices out of the range bound from (2) and (3) by choosing the volatility parameter.

4.2 Lower Bound from the (Li 2009) Model

More precise lower bound can be obtained if the $\mathcal{Y}_t$ in (3) also includes single name information. To study the effects of single name information, we used the model described in (Li 2009). The (Li 2009) model is a one-factor bottom-up dynamic model where the systemic factor is modeled by an increasing process X_t. Under the (Li 2009) model, the marginal distribution of X_t determines the *JDDI*; the Markov chain on X_t determines the *JDDT*, therefore each different Markov chain on X_t defines a different default time copula. The marginal distribution of X_t can be calibrated to index tranche prices, afterwards, we can construct different default time copulas by constructing different Markov chain to the marginal distribution of X_t. These different default time copula produces different *JDDT* but identical *JDDI* and tranche prices by construction.

Assume $\mathcal{S}_t$ is a a filtration generated by the common factor process X_t and the single name default and recovery. Of course $\mathcal{S}_t \subset \mathcal{F}_t$ since $\mathcal{F}_t$ includes other systemic and idiosyncratic factors beyond X_t. In the numerical implementation, the lower bound from $\mathcal{S}_t$ are computed by only conditioning on the value of the common factor X_t but not the loss L_t, which allow us to compute the lower bound with a semi-analytical method pioneered by (Andersen, Sidenius & Basu 2003). Ignoring the realized loss L_t results in slightly worse (or lower) lower bounds because we are not using the full information available in $\mathcal{S}_t$, but it is a good trade off since it allows us to use the semi-analytical pricing method for much faster calculation of the lower bound. Since the X_t and the loss L_t are highly correlated under the one-factor model, the degradation of the lower bound quality is expected to be small by excluding the realized losses in the conditioning.

Table 4: Lower Bounds of 3Y-5Y Option from $\mathcal{S}_t$

CDX-IG9	ITM Lower Bounds			ATM Lower Bounds			OTM Lower Bounds		
Tranches	Co-mo	Max-E	LLB	Co-mo	Max-E	LLB	Co-mo	Max-E	LLB
0-3%	41.07%	40.93%	40.05%	10.47%	9.53%	8.46%	0.00%	0.00%	0.00%
3-7%	29.15%	26.63%	22.07%	19.13%	15.45%	11.96%	3.19%	2.57%	2.47%
7-10%	16.26%	14.48%	13.12%	13.20%	11.74%	11.20%	8.12%	7.85%	7.58%
10-15%	7.32%	7.26%	6.42%	6.70%	6.64%	5.79%	5.54%	5.47%	4.85%
15-30%	2.19%	2.14%	1.23%	2.12%	2.04%	1.16%	1.99%	1.90%	1.09%
30-60%	1.60%	1.53%	0.82%	1.57%	1.47%	0.65%	1.52%	1.37%	0.73%
60-100%	0.41%	0.39%	0.20%	0.41%	0.38%	0.12%	0.41%	0.35%	0.14%

In this example, we built a co-monotonic Markov chain and a maximum entropy Markov chain on X_t, and the resulting lower bounds from $\mathcal{S}_t$ and these two default time copula are show in table 4. We can also find the lowest lower bound of all possible Markov chains of X_t that preserves the *JDDI* and tranche prices. We denote the lowest lower bound based on the sub-filtration $\mathcal{S}_t$ as LLB($\mathcal{S}_t$) (shown in the column "LLB" of table 4). The LLB($\mathcal{S}_t$) is much higher than the LLB($\mathcal{L}_t$) because it is constrained by the additional single name information and the conditional independent correlation structure in the (Li 2009) model.

Comparing table 3 and 4, it is interesting to note that the lower bounds from $\mathcal{L}_t$ can vary at a much wider range than the lower bound from $\mathcal{S}_t$. This can be explained by the fact that not all the loss transitions are admissible under a factor model with conditional independence. For example, the co-monotonic Markov chain built on the loss process have many deterministic transitions, such as: if the tranche loss is 2% at 3Y, then the loss will be 4% at 5Y with probability 1. The existence of such fully deterministic transition is a property of the co-monotonic Markov chain. Though admissible under a contagion model, the deterministic loss transition is incompatible with a conditional independent factor model where the L_T conditioned on $\mathcal{F}_t$ can never be fully deterministic except for the degenerated case. Therefore, adding single name information and a conditional independent correlation structure further restricts the set of admissible loss transitions, thus imposing a narrower range on the lower bounds.

(Lando & Nielsen 2009) have shown that the conditional independent assumption cannot be rejected from either the individual case studies or the statistical tests of the historical default events. None of the historical default events so far are caused by contagion in the strict sense that one company's default event directly caused another company to default. Contagion models also have some undesirable properties as shown in (Hitier & Huber 2009) that make it difficult to use in practice. Therefore, conditional independent factor model remains the most practical and efficient approach to include single name information. In practice, we have to adopt the lower bounds from the conditional independent model since it is the only feasible approach to price and manage both the vanilla CDO tranches and exotic instruments like tranche options.

4.3 Systemic vs. Idiosyncratic Dynamics

Furthermore, we can quantify how much uncertainty of the option value is due to systemic dynamics vs. idiosyncratic dynamics under the (Li 2009) model. Suppose we have a filtration $\mathcal{U}_t$ which include $\mathcal{S}_t$ and X_T, i.e., this filtration correspond to a less powerful deity (comparing to the all-powerful deity that gives the upper bound) who can only foresee the future value of the common factor, but not the idiosyncratic default events. The remaining uncertainty between the lower bound from $\mathcal{U}_t$ and the upper bound has to be caused by idiosyncratic dynamics. Therefore, the option bounds from $\mathcal{U}_t$ gives a way to gauge the pricing uncertainty purely due to the idiosyncratic dynamics. Table 5 showed the lower bounds calculated

Table 5: Lower Bounds of 3Y-5Y Option from $\mathcal{U}_t$ (Perfect Foresight)

CDX-IG9 Tranches	Lower Bounds		
	ITM	ATM	OTM
0-3%	41.14%	10.73%	0.00%
3-7%	29.47%	19.15%	3.28%
7-10%	16.37%	13.38%	8.18%
10-15%	7.36%	6.70%	5.69%
15-30%	2.20%	2.13%	2.04%
30-60%	1.60%	1.59%	1.56%
60-100%	0.42%	0.41%	0.41%

from $\mathcal{U}_t$. By comparing to the upper bounds in Table 2, it is obvious that option pricing uncertainty due to idiosyncratic dynamics is very limited. The idiosyncratic dynamics only contributes a small amount of uncertainty to junior tranche options, and it has almost no contributions to the senior tranche options. Therefore, we can safely ignore the idiosyncratic spread dynamics if we are mainly dealing with the senior tranche options.

4.4 Long-dated Options

The upper and lower bounds of a 5Y to 10Y tranche loss option are also computed in Table 6. In general, the price bounds of the 5Y-10Y options exhibit very similar features as the 3Y-5Y options. The 5Y-10Y option showed a wider range between upper and lower bound than the 3Y-5Y option, which is not surprising since the long-dated option is expected to have more pricing uncertainties.

4.5 Choice of Markov Chains

In Table 5, the lower bounds from $\mathcal{U}_t$ is only slightly higher than the lower bound from co-monotonic Markov chain because the co-monotonic Markov chain is very close to having perfect foresight as the common factors at the two maturities are mapped sequentially by their distribution quantiles. If the common factor is specified as a continuous distribution, the co-monotonic Markov chain will produce the exact same lower bound as those from $\mathcal{U}_t$. Therefore, it is arguable that the co-monotonic Markov chain is not realistic due to the collapsed uncertainty of future common factor distribution.

As noted by many previous authors, eg (Andersen 2006) and (Skarke 2005), the classic Gaussian Copula implies very unrealistic spread dynamics. This price bound analysis of tranche options offers yet another interesting view on the Gaussian Copula: the classic Gaussian Copula model is a degenerated co-monotonic Markov chain across time where the common factor distributions remains unchanged (Gaussian). Therefore, the classic Gaussian copula suffers from the same problem of vanishing common factor uncertainties as the co-monotonic Markov chain. Co-monotonic Markov chains, including Gaussian Copula, will overvalue the tranche options because of the perfect foresight of the future market factor realizations.

In comparison, the maximum entropy Markov chain is a much better choice since it has the advantage of keeping the least amount of information in the system, and the uncertainty of the future common factor is the largest among all possible Markov chains (because the information entropy is maximized). Ideally we should calibrate the Markov chain using market information; however it is impossible to do so under the current market condition because there is no relevant and reliable market observables on the transition of future loss process. Given the lack of market information, we argue that the maximum entropy Markov chain

Table 6: Price Bounds of 5Y-10Y Tranche Loss Option

In-the-Money Option

CDX-IG9	Lower Bounds					Upper
Tranches	LLB($\mathcal{L}_t$)	Max-E $\mathcal{L}_t$	LLB($\mathcal{S}_t$)	Max-E $\mathcal{S}_t$	$\mathcal{U}_t$	Bound
0-3%	45.46%	45.58%	45.37%	45.56%	45.40%	46.97%
3-7%	33.19%	35.75%	35.76%	39.64%	41.28%	42.12%
7-10%	23.23%	26.75%	25.15%	30.90%	33.13%	34.66%
10-15%	11.94%	14.25%	11.81%	16.47%	18.07%	19.29%
15-30%	3.91%	5.08%	3.96%	6.58%	7.22%	7.42%
30-60%	2.61%	3.49%	2.62%	4.60%	5.14%	5.14%
60-100%	0.88%	1.16%	0.76%	1.37%	1.50%	1.50%

At-the-Money Option

CDX-IG9	Lower Bounds					Upper
Tranches	LLB($\mathcal{L}_t$)	Max-E $\mathcal{L}_t$	LLB($\mathcal{S}_t$)	Max-E $\mathcal{S}_t$	$\mathcal{U}_t$	Bounds
0-3%	5.61%	5.76%	4.96%	5.62%	5.62%	7.09%
3-7%	12.91%	14.92%	15.80%	17.98%	19.99%	20.53%
7-10%	8.16%	13.70%	11.82%	16.85%	18.49%	22.23%
10-15%	5.60%	8.87%	9.27%	12.06%	14.33%	15.69%
15-30%	1.98%	3.94%	3.49%	5.96%	6.75%	6.93%
30-60%	1.52%	2.91%	1.51%	4.27%	4.97%	4.98%
60-100%	0.46%	1.01%	0.56%	1.27%	1.47%	1.47%

Out-of-the-Money Option

CDX-IG9	Lower Bounds					Upper
Tranches	LLB($\mathcal{L}_t$)	Max-E $\mathcal{L}_t$	LLB($\mathcal{S}_t$)	Max-E $\mathcal{S}_t$	$\mathcal{U}_t$	Bound
0-3%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3-7%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
7-10%	0.53%	0.53%	0.52%	0.50%	0.80%	1.36%
10-15%	3.23%	4.57%	5.57%	6.47%	8.15%	9.65%
15-30%	1.69%	2.98%	2.90%	5.00%	5.92%	6.08%
30-60%	1.43%	2.33%	1.41%	3.70%	4.65%	4.66%
60-100%	0.44%	0.86%	0.54%	1.15%	1.41%	1.41%

Table 7: 3Y-5Y ATM Options with Single Default Event Trigger

Trigger Default Prob = 5%

CDX-IG9 Tranches	Independent		Less Correlated		More Correlated	
	LB	UB	LB	UB	LB	UB
0-3%	0.48%	0.64%	0.69%	0.73%	0.86%	0.87%
3-7%	0.77%	1.03%	1.57%	1.69%	2.14%	2.22%
7-10%	0.59%	0.74%	1.69%	1.75%	2.41%	2.46%
10-15%	0.33%	0.36%	1.37%	1.37%	1.89%	1.90%
15-30%	0.10%	0.11%	0.89%	0.90%	1.09%	1.11%
30-60%	0.07%	0.08%	0.80%	0.81%	0.94%	0.96%
60-100%	0.02%	0.02%	0.22%	0.22%	0.25%	0.26%

Trigger Default Prob = 30%

CDX-IG9 Tranches	Independent		Less Correlated		More Correlated	
	LB	UB	LB	UB	LB	UB
0-3%	2.86%	3.82%	3.92%	4.15%	4.72%	4.88%
3-7%	4.63%	6.17%	8.19%	9.02%	10.47%	11.17%
7-10%	3.52%	4.44%	7.79%	8.22%	9.95%	10.40%
10-15%	1.99%	2.13%	5.00%	5.02%	6.09%	6.11%
15-30%	0.61%	0.66%	1.82%	1.88%	2.00%	2.07%
30-60%	0.44%	0.48%	1.36%	1.44%	1.45%	1.56%
60-100%	0.11%	0.12%	0.36%	0.38%	0.38%	0.40%

is the most natural choice since it corresponds to the state of no relevant information. Table 3 to 6 have shown that the lower bounds become more precise with larger sub-filtrations, and the price bounds imposed by $\mathcal{S}_t$ and the maximum entropy Markov chains are still quite tight, especially for the senior tranches. Therefore, the dynamics of other systemic and idiosyncratic factors beyond $\mathcal{S}_t$ only have limited contribution to the pricing uncertainty of tranche options.

5 Tranche Options with Random Triggers

As discussed in section 2, the trigger event itself for a tranche option can be a random event. In general, the tranche options with random triggers require Monte Carlo simulation to compute its upper and lower bounds. However, if the trigger event is the default of a single credit, we can still treat it semi-analytically without Monte Carlo simulation by taking advantage of the conditional independence.

5.1 Single Default Event Trigger

We take the 3Y to 5Y CDX-IG9 ATM tranche loss option as an example, and consider an option that can be exercised at 3Y only if a trigger credit has defaulted before 3Y. We further assume that the trigger credit does not appear in the portfolio of the CDO tranche[2]. Because of the single credit trigger, the price bounds of this option cannot be obtained using a pure top-down model.

[2] If the name does appear in the tranche portfolio, we can always replicate the original option by an equivalent tranche loss option without the trigger credit in the portfolio by adjusting the tranche attachment, detachment and the strike price of the option because the trigger credit has to be in the default state when the option has non-zero payoff.

We consider the price bound with a Maximum Entropy Markov chain under the (Li 2009) model. Because of the conditional independence, whether the trigger name default before 3Y does not change the distribution or the transition Markov Chain of the common factor process; neither does it change the conditional default probabilities of any other names in the portfolio. Therefore, we can obtain the price bounds of this option by simply weighting the option payoff in (2) and (3) by the 3Y conditional default probability of the trigger credit:

$$\begin{aligned} C^U &= \mathbb{E}[\mathbf{1}_{\tau<t} \max(\sum_{t_i>t} d(0,t_i)c_i - K, 0)] \\ &= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\tau<t} \max(\sum_{t_i>t} d(0,t_i)c_i - K, 0)|X_t]] && \text{: Iterative expectation} \\ &= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\tau<t}|X_t]\mathbb{E}[\max(\sum_{t_i>t} d(0,t_i)c_i - K, 0)|X_t]] && \text{: Conditional Independence} \\ &= \mathbb{E}[q(X_t,t)\mathbb{E}[\max(\sum_{t_i>t} d(0,t_i)c_i - K, 0)|X_t]] \end{aligned} \tag{8}$$

Where X_t is the value of the common factor at time t, $q(X_t,t)$ is the conditional default probability of the trigger credit. Similarly, we can get the following expression for the lower bound:

$$C^L = \mathbb{E}[q(X_t,t)\mathbb{E}[\max(\mathbb{E}[\sum_{t_i>t} d(0,t_i)c_i|\mathcal{Y}_t] - K, 0)|X_t]] \tag{9}$$

Both of the bounds in (8) and (9) are easy to compute semi-analytically.

As shown in Table 7, we computed the price bounds with two different default probabilities for the trigger credit, 5% and 30%. We also computed the price bounds with different correlations between the trigger credit default and the common market factor. As expected, the option is more valuable if the trigger credit is more risky. Table 7 also showed that the option is much more valuable if the trigger credit is more correlated to the common market factor. In the context of counterparty risk, this results in the so called "wrong-way" risk of buying tranche protection from a risky counterparty, i.e., the tranche protection worth much less if the counterparty is more likely to default when the portfolio suffers more losses.

The price bounds in Table 7 are much narrower than the comparable bounds from Maximum Entropy Markov chain in table 2 and 4. The price bounds are very narrow even when the trigger name has significant default probability; therefore there is clearly no need to build the full dynamic spread models for the single default event trigger. Counterparty risk of tranches therefore can be very effectively priced and managed using this methodology.

In this example, we assume the option is exercised at 3Y if the trigger name defaults before 3Y, we refer to it as the "exercise-at-maturity" option. In a more realistic setting, the option holder has to exercise the option immediately if the trigger credit default, which is referred as "exercise-at-trigger". The "exercise-at-trigger" option can be modeled as a series "exercise-at-maturity" options, with one option expires at every default observation date and is only exercisable if the trigger credit defaults between the previous default observation date and the current default observation date. Given that he "exercise-at-maturity" option expires at a fixed maturity date and the "exercise-at-trigger" option is a series of options that expires at each default observation date from time 0 to the maturity, the "exercise-at-trigger" option's value is always less than the "exercise-at-maturity" option with the same maturity because an option is less valuable with shorter expiration.

Table 8: Max Entropy Lower Bounds Conditioned on X_t and L_t

CDX-IG9 Tranches	3Y-5Y Lower Bounds ITM	ATM	OTM	5Y-10Y Lower Bounds ITM	ATM	OTM
0-3%	41.39%	10.31%	0.00%	46.14%	6.25%	0.00%
3-7%	26.74%	15.68%	2.72%	40.03%	18.14%	0.00%
7-10%	14.45%	11.86%	8.08%	30.94%	17.27%	0.67%
10-15%	7.23%	6.62%	5.54%	16.64%	12.38%	6.93%
15-30%	2.12%	2.02%	1.88%	6.55%	5.95%	4.97%
30-60%	1.50%	1.45%	1.34%	4.54%	4.22%	3.66%
60-100%	0.39%	0.38%	0.35%	1.35%	1.26%	1.14%

5.2 Generic Random Triggers

For more general triggers that involve multiple names, such as portfolio loss triggers or the 1st default event in a credit basket, we have to use Monte Carlo simulation of default time and recovery to compute the price bounds. The semi-analytical solutions for these complicated triggers often gets too tedious comparing to the straight-forward Monte Carlo simulation.

The lower bound of the option depends on the term $\mathbb{E}[V_t|\mathcal{Y}_t]$ in (3). In section 4, we restricted ourselves to only condition on the common factor X_t, which is the most convenient for semi-analytical solutions. However, in Monte Carlo simulation, we can easily add additional variables in the filtration $\mathcal{Y}_t$ to the conditioning so that we can get better (or higher) lower bounds. The realized portfolio loss L_t is the next most useful factor to be included in the conditioning after the common factor X_t. Given the conditional independence, there is limited benefits to include individual names' default indicators in the conditioning after X_t and L_t. If we only use X_t, L_t as the two conditioning variables, the $\mathbb{E}[V_t|X_t, L_t]$ can be directly computed from the simulation by constructing a two dimensional grid that samples the X_t and L_t discretely. If there are more variables in the conditioning, we have to use the regression technique in the typical least square Monte Carlo methodology in (Longstaff & Schwartz 2001).

Table 8 shows the lower bound of the tranche loss option with deterministic time trigger implied by the maximum entropy Markov chain conditioned on both X_t and L_t. The results are obtained from a Monte Carlo simulation where half of the simulated path is used to establish the $\mathbb{E}[V_t|X_t, L_t]$ by constructing a two-dimensional grid of (X_t, L_t), and the other half of the simulated path is used to compute the actual lower bounds from the $\mathbb{E}[V_t|X_t, L_t]$. Comparing with Table 4 and Table 6, the lower bound for junior tranches improved slightly by adding the realized loss L_t in the conditioning. The lower bounds of senior tranches showed almost no improvements.

We also considered a more realistic example of callable tranche where a client sold protection to a bank on a senior 5Y IG9 tranche, and the client has the right to buy back the protection at the initial expected tranche loss if the IG9 portfolio loss is greater than a pre-determined threshold α at the 3Y. In this example, the trigger event and the option payoff are highly correlated as both of them are functions of the IG9 portfolio loss, therefore, we cannot compute its price bounds by simply multiplying the tranche option payoffs in Table 4 by the probability of the trigger event. Instead we have to use the full Monte Carlo simulation to compute the price bounds, which are shown in Table 9. The price bounds of the options to call tranche with portfolio loss triggers are also very tight.

The price bounds of other types of options, such as gap risk and liquidation risk, can also be computed from Monte Carlo simulation of default times and recovery rate using similar methods as in the callable tranche.

Table 9: Price Bounds of 3Y-5Y Option to Call Tranche

CDX-IG9	$\alpha = 4\%$		$\alpha = 8\%$		$\alpha = 12\%$	
Tranches	LB	UB	LB	UB	LB	UB
15-30%	2.02%	2.22%	1.85%	1.95%	1.08%	1.13%
30-60%	1.45%	1.62%	1.36%	1.47%	0.95%	1.00%
60-100%	0.38%	0.42%	0.36%	0.38%	0.26%	0.27%

6 Conclusion

In this study, we have shown that the tranche option prices can be effectively bounded from a default time copula. We argue that the default time copula from the maximum entropy Markov chain is the most natural choice when we don't have relevant market observables for tranche options; we also argue that it is possible for a dealer to make market and dynamically hedge the senior tranche options solely based on their price bounds.

For the European tranche options and the tranche options with single name default triggers, both the upper bound and lower bound can be computed from semi-analytical methods without Monte Carlo simulation. The Greeks of the price bounds can be computed by perturbing the market inputs and re-valuing. When the pricing bounds of an option is narrow, a dealer can treat the true value the option as an average of the lower and upper bound, and dynamically hedge an European tranche option book by averaging the Greeks for the lower bound and upper bound. A senior tranche option book could be managed without much additional effort than the effort required to risk-manage a typical index or bespoke tranche book.

The fact that the upper and lower bounds can be hedged using static instruments makes it possible to profit from the market mis-pricing of the tranche options. In the event that the market tranche option prices are out of the pricing bounds, the dealer can take the corresponding option position and a hedge position for the bound that the option value violates. For example, if a European tranche call option on a A to D tranche with strike K is priced higher in the market than its upper bound, then a dealer can sell the call option and hedge it by buying protection on a $A+K$ to D tranche. This will results in a positive profit without any future risk, i.e., an arbitrage opportunity. However, in reality, the $A+K$ to D tranche may not be liquid itself, therefore, the tranche option may have to be hedge dynamically using other liquid instruments. The dynamic hedging of the lower or upper bound of the tranche options is no more complicated than the common practice of dynamically hedging the off-the-run index tranches or bespoke tranches.

Contrary to a common stereotype, the tranche options prices are primarily determined by the *JDDT* and default time copula. Other systemic and idiosyncratic factors beyond the default time copula contribute a relatively small portion of the pricing uncertainty. Using top-down models to price tranche options could result in mis-pricing given the top-down models ignores a large amount of static single name market information which is important in determining the tranche option prices.

The methodology to obtain price bounds of tranche options described in this paper could also play an important role in the management of counterparty risk, gap risk and liquidation risk. As shown in section 5.2, the price bounds are often very narrow for tranche options with random trigger event, therefore, these practical problems can also be effectively addressed using the price bounds.

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Appendix

A Approximate the Tranche PV Option

Consider a generic tranche with non-zero coupon, and whose protection payment is settled at the time of default. The MTM of such a tranche can be written as:

$$V_t = PROT_t - sPV01_t$$

where $PROT$ is the protection PV and the $PV01$ is the PV of a 1bps coupon payment, and s is the contractual coupon. Since the main risk factor of a tranche is its terminal loss, we can view the $PROT_t$ and $PV01_t$ as functions of the expected terminal tranche loss $l_t = \mathbb{E}[L_T(A,D)|\mathcal{F}_t]$. Therefore, we can expand the $PROT(l_t)$ and $PV01(l_t)$ around the current tranche expected loss $l_t^0 = \mathbb{E}[L_T(A,D)|\mathcal{F}_0]$:

$$PROT(l_t) \approx PROT(l_t^0) + \frac{\partial PROT(l_t^0)}{\partial l_t}(l_t - l_t^0)$$
$$PV01(l_t) \approx PV01(l_t^0) + \frac{\partial PV01(l_t^0)}{\partial l_t}(l_t - l_t^0)$$

Therefore, we have:

$$\begin{aligned} V_t &\approx PROT(l_t^0) - sPV01(l_t^0) + \left(\frac{\partial PROT(l_t^0)}{\partial l_t} - s\frac{\partial PV01(l_t^0)}{\partial l_t}\right)(l_t - l_t^0) \\ &= a + bl_t \end{aligned}$$

Where a, b are constants which are obvious from the above equation. This effectively approximates the MTM of a regular tranche by a linear function of l_t. The first order derivatives can be obtained from the default time copula and the price of an option on tranche PV with strike K can be approximated by a tranche loss option:

$$\begin{aligned} C &= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t - K, 0)] \\ &\approx \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(a + bl_t - K, 0)] \\ &= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t} b \max(l_t - \frac{K-a}{b}, 0)] \end{aligned}$$

Therefore, the tranche loss option bounds discussed in this paper can be applied to produce the bounds for the tranche PV option. In practice, the tranche PV is mainly driven by its terminal loss, thus the approximation should be adequate in most situations.

B Finding the Lowest Lower Bound

We want to find the lowest lower bound among all possible Markov chains from two loss (or common factor) distributions; this problem can be formulated as an nonlinear optimization with linear constraints. Here we use the LLB for the loss distribution to describe the optimization setup, the LLB of the common factor process can be solved using the exact same method.

Suppose the portfolio loss distribution at two time horizons $t < T$ are sampled by a discrete loss grid l_i, the corresponding loss probability are given by $p_i(t)$ and $p_i(T)$, which has to satisfy the usual constraints of being valid loss distributions.

A discrete Markov chain is parameterized by its transition probability $q_{ij} = \mathbb{P}\{l_j|l_i\}$, obviously q_{ij} is zero if $i > j$. The transition probability has to satisfy the following constraints from the initial loss distribution:

$$\begin{aligned} \sum_i q_{ij} &= p_i(t) \\ \sum_j q_{ij} &= p_j(T) \end{aligned} \tag{10}$$

The lower bound from the Markov chain q_{ij} is a nonlinear objective function that we have to minimize by adjusting the q_{ij}. Therefore, this is a nonlinear optimization problem with linear constraints given in (10). The dimension of $q_{i,j}$ is about $N^2/2$ where N is the number of discrete samples on the loss distribution. For a N less than 40, this problem can be solved in a few minutes using Powell's TOLMIN algorithm.